Lecture IV

Stationary Stokes and Navier-Stokes Equations with Different Physical Boundary Conditions

Outline

Introduction and motivation

We are interested by the following Stokes equations:

$$
\begin{aligned}
-\Delta\, \boldsymbol{u} + \nabla\, \pi &= \boldsymbol{f} \qquad \text{in } \Omega, \\
\operatorname{div} \boldsymbol{u} &= 0 \qquad \text{in } \Omega,
\end{aligned}
$$

with the following nonhomogeneous boundary conditions:

$$
\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \;\; \text{and} \;\; \pi = \pi_0 \qquad \text{on } \Gamma, \tag{1}
$$

or

$$
\boldsymbol{u} \cdot \boldsymbol{n} = g \;\; \text{and} \;\; \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad \text{on } \Gamma, \tag{2}
$$

or the following Navier boundary condition

$$
\boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h}, \tag{3}
$$

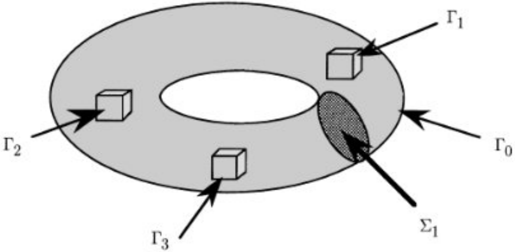
We will study also here the case of the Navier-Stokes equations:

Find $\boldsymbol{u}$, π, $\alpha_1, \dots, \alpha_I$, with $\alpha_i \in \mathbb{R}$

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i, \; i = 1, \dots, I, \end{cases}$$

where we suppose that Ω is an open set possibly multiply connected sufficiently regular with a boundary Γ possibly non connected. We denote $\Gamma = \bigcup_{i=0}^{I} \Gamma_i$ with Γ_i the connected components of Γ and $\Sigma = \bigcup_{j=1}^{J} \Sigma_j$ and Σ_j a finite number of cuts.

$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$ is simply connected.



Considering for example the case of the Stokes equations with the homogeneous boundary conditions

$$(\mathcal{S}_T^0) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \end{cases}$$

Because these boundary conditions, we write

$$-\Delta\, \boldsymbol{u} = \mathbf{curl}\, \mathbf{curl} \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u}$$

For the variational formulation, we will consider the following spaces:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{v} = 0,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},$$

We will prove later that the Stokes problem $(\mathcal{S}_T^0)$, with $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\boldsymbol{u} \in \boldsymbol{V}$ and $\pi \in L^2(\Omega)$, is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V} \text{ such that} \\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}. \end{cases}$$

Questions:

- Because $\boldsymbol{u}$ is apparently only in $\boldsymbol{H}^1(\Omega)$, how to give a sense to the following boundary condition

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma\,?$$

- The bilinear form is it coercive to apply the Lax-Milgram lemma ?

  We know (see Lecture I) that if Ω is simply connected, we have:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

- What happens if Ω is not simply connected ?
- Can we find generalized solution in $\boldsymbol{W}^{1,p}(\Omega)$ with $1 < p < \infty$?
- Can we find strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ with $1 < p < \infty$?
- Can we find very weak solution in $\boldsymbol{L}^p(\Omega)$ with $1 < p < \infty$?

II. Stokes problems with normal boundary conditions

Consider the following Stokes problem:

$$(\mathcal{S}_T)\quad \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = g,\ \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, & 1\le j\le J. \end{cases}$$

Lemma 2.1

Suppose that $\boldsymbol{\psi}\in \boldsymbol{W}^{1,p}(\Omega)$. Then

$$\mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n} = \operatorname{div}_\Gamma(\boldsymbol{\psi}\times\boldsymbol{n}) \quad \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Proof. To simplify, suppose $p = 2$. For any $\chi\in H^2(\Omega)$, Green formulas yield

$$\begin{aligned} \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi &= \langle \mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n},\,\chi\rangle_\Gamma\,, \\ \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi &= -\langle \boldsymbol{\psi}\times\boldsymbol{n},\,\mathbf{grad}\,\chi\rangle_\Gamma \\ &= \langle \operatorname{div}_\Gamma(\boldsymbol{\psi}\times\boldsymbol{n}),\,\chi\rangle_\Gamma\,. \end{aligned}$$

II. Stokes problems with normal boundary conditions

Consider the following Stokes problem:

$$(\mathcal{S}_T) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

Lemma 2.1

Suppose that $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$. Then

$$\mathbf{curl}\, \boldsymbol{\psi} \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(\boldsymbol{\psi} \times \boldsymbol{n}) \quad \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Proof. To simplify, suppose $p = 2$. For any $\chi \in H^2(\Omega)$, Green formulas yield

$$\begin{aligned} \int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{grad}\, \chi &= \langle \mathbf{curl}\, \boldsymbol{\psi} \cdot \boldsymbol{n},\, \chi \rangle_\Gamma\,, \\ \int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{grad}\, \chi &= -\langle \boldsymbol{\psi} \times \boldsymbol{n},\, \mathbf{grad}\, \chi \rangle_\Gamma \\ &= \langle \operatorname{div}_\Gamma(\boldsymbol{\psi} \times \boldsymbol{n}),\, \chi \rangle_\Gamma\,. \end{aligned}$$

Applying the divergence operator in Problem $(\mathcal{S}_T)$, we get firstly

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega.$$

Setting then $\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u}$, we have

$$-\Delta\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{in } \Omega$$

and

$$-\Delta\,\boldsymbol{u}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n} = (\boldsymbol{f} - \nabla\,\pi)\cdot\boldsymbol{n}.$$

So formally the pressure satisfies the following Neumann boundary condition:

$$\frac{\partial\,\pi}{\partial\,\boldsymbol{n}} = \boldsymbol{f}\cdot\boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \quad \text{on } \Gamma$$

So, we can solve the pressure directly in the Stokes problem $(\mathcal{S}_T)$.

Let us introduce the following space:

$$\boldsymbol{H}_0^{r,p}(\mathrm{div},\,\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^r(\Omega);\ \ \mathrm{div}\,\boldsymbol{\varphi} \in L^p(\Omega),\ \ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\},$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,p}(\mathrm{div},\,\Omega)} = \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

We can prove that

$$\boldsymbol{D}(\Omega) \quad \text{is dense in} \quad \boldsymbol{H}_0^{r,p}(\mathrm{div},\,\Omega).$$

So its dual is then a subspace of $\boldsymbol{D}'(\Omega)$ which can be characterized as:

$$[\boldsymbol{H}_0^{r,p}(\mathrm{div},\,\Omega)]' = \{\boldsymbol{F} + \mathbf{grad}\,\psi;\ \ \boldsymbol{F} \in \boldsymbol{L}^{r'}(\Omega),\ \psi \in L^{p'}(\Omega)\}.$$

Lemma 2.2

Suppose that

$$\boldsymbol{z} \in [\boldsymbol{H}_0^{6,\,2}(\mathrm{div},\,\Omega)]',$$

that means that

$$\boldsymbol{z} = \nabla\pi - \boldsymbol{f}, \quad \text{with} \quad \pi \in L^2(\Omega) \qquad \text{and } \boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$$

and assume div $\boldsymbol{z} = 0$ in Ω. Then

$$\boldsymbol{z} \cdot \boldsymbol{n} \in H^{-3/2}(\Gamma)$$

and for any $\chi \in H^2(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$, we have

$$\langle \boldsymbol{z}, \nabla\chi \rangle_{[\boldsymbol{H}_0^{6,\,2}(\mathrm{div},\,\Omega)]' \times \boldsymbol{H}_0^{6,\,2}(\mathrm{div},\,\Omega)} = \langle \boldsymbol{z} \cdot \boldsymbol{n},\, \chi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}$$

Proposition 2.3

For any

$$\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\Gamma)$$

there exists $\pi \in L^2(\Omega)$, unique up an additive constant, such that

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega, \quad \frac{\partial\,\pi}{\partial\,\boldsymbol{n}} = \boldsymbol{f}\cdot\boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \quad \text{on } \Gamma \qquad (4)$$

Proof.

Problem (4) is equivalent to the following very weak formulation: for any $\chi \in H^2(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$

$$\int_\Omega \pi\Delta\chi = -\int_\Omega \boldsymbol{f}\cdot\nabla\chi + \langle \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}),\, \chi\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}$$

that we solve by duality thanks to the H^2-regularity for the strong Neumann problem with the RHS in $L^2(\Omega)$.

- To solve the Stokes problem $(\mathcal{S}_T)$, without loss generality, we suppose that $g = 0$.

- We consider here only the hilbertian case: we search the velocity in $H^1(\Omega)$ and the pressure in $L^2(\Omega)$. For that, we will suppose that

$$\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\Gamma).$$

- We solve first the following Neumann problem:

  There exists a very weak solution $\pi \in \boldsymbol{L}^2(\Omega)$, unique up an additive constant, satisfying:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega, \quad (\nabla\,\pi - \boldsymbol{f}) \cdot \boldsymbol{n} = -\operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) \quad \text{on } \Gamma$$

Remark

- Unlike the case of the Stokes problem with Dirichlet boundary condition, it appears that when

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f} \cdot \boldsymbol{n} - \operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) = 0 \quad \text{on } \Gamma$$

  the pressure π can be constant.

Setting

$$\boldsymbol{H} = \boldsymbol{H}_0^{6,2}(\mathrm{div},\, \Omega)$$

and let us consider the following space

$$\boldsymbol{E}(\Delta, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\, \Delta\, \boldsymbol{v} \in \boldsymbol{H}'\},$$

which is a Banach space for the graph norm.

We have the following preliminary results:

$$D(\overline{\Omega}) \quad \text{is dense in } \boldsymbol{E}(\Delta, \Omega).$$

As a consequence, we have the following result.

Proposition 2.4

The linear mapping $\gamma : \boldsymbol{v} \to \mathbf{curl}\,\boldsymbol{v}|_\Gamma \times \boldsymbol{n}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping

$$\gamma : \boldsymbol{E}(\Delta, \Omega) \longrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}(\Delta, \Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^1_T(\Omega)$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω,

$$-\langle \Delta\,\boldsymbol{v},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}'\times\boldsymbol{H}} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma, \quad (5)$$

where the duality on Γ is defined by $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}$.

Proposition 2.5 (Weak and Strong solutions of (S_T))

i) Let $g = 0$,

$$\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega), \quad \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\Gamma),$$

satisfying the following compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}^2_T(\Omega), \ \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{h} \times \boldsymbol{n}, \, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \ \boldsymbol{H}^{1/2}(\Gamma)} = 0.$$

Then, the problem (S_T) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ satisying the estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\| \boldsymbol{f} \|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}).$$

ii) If moreover Ω is of class $\mathcal{C}^{2,1}$ and $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1/6,6/5}(\Gamma)$, then the solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{2,6/5}(\Omega) \times W^{1,6/5}(\Omega)$. If $f \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$, then the solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$.

Proof.

- Observe first that if $\boldsymbol{u} \in \boldsymbol{H}^{1}(\Omega)$ is solution of Problem S_T, then $\Delta \boldsymbol{u} \in \boldsymbol{H}'$ and then

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\Gamma).$$

  So the boundary condition of the tangential component of the vorticity of $\boldsymbol{u}$ has a sense.

To prove the existence of weak solution, we will use Lax-Milgram Lemma.

It is easy to see that if $\boldsymbol{u} \in \boldsymbol{V}$ is solution of Problem (S_T), then

$$(\mathcal{P}_T^0) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \ \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \langle \boldsymbol{h} \times \boldsymbol{n}, \, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \, \boldsymbol{H}^{1/2}(\Gamma)}.$$

where we recall that

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \, \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \, \mathrm{div}\, \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ and } \langle \boldsymbol{v} \cdot \boldsymbol{n}, \, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J\},$$

and

$$\boldsymbol{V} \hookrightarrow \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega).$$

(observe the compatibility condition)

- In fact, Problem $(\mathcal{P}_T^0)$ is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$(\mathcal{Q}_T^0) \quad \forall \boldsymbol{w} \in \boldsymbol{W}, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \langle \boldsymbol{h} \times \boldsymbol{n}, \, \boldsymbol{v} \rangle_\Gamma$$

where

$$\boldsymbol{W} = \{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega), \, \mathrm{div}\, \boldsymbol{v} = 0, \; \mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^2(\Omega), \;\; \boldsymbol{w} \cdot \boldsymbol{n} = 0 \}.$$

- Taking $\boldsymbol{w} \in \mathcal{D}(\Omega)$, with div $\boldsymbol{w} = 0$, then by de Rham's Theorem we deduce that there exits $\pi \in L^2(\Omega)$ such that

$$-\Delta\, \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in } \Omega.$$

- Next multiplying this equation by $\boldsymbol{w} \in \boldsymbol{W}$ and using Green-Formula, we deduce that

$$\forall \boldsymbol{w} \in \boldsymbol{W}, \quad \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n},\, \boldsymbol{w} \rangle_\Gamma = \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{w} \rangle_\Gamma.$$

Now, for any $\boldsymbol{\mu} \in \boldsymbol{H}^{1/2}(\Gamma)$, there exists

$$\boldsymbol{w} \in \boldsymbol{W} \quad \text{with } \boldsymbol{w} = \boldsymbol{\mu}_\tau \quad \text{on } \Gamma.$$

Consequently

$$\langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma = \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma.$$

III. Stokes Equations with Pressure Boundary Condition

Here, we decompose the Stokes problem in two problems

$$(\mathcal{S}_N^0) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

and

$$(\mathcal{S}_N^1) \quad \begin{cases} -\Delta\, \boldsymbol{w} + \nabla\, \theta = \boldsymbol{0} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, \quad \theta = \theta_0 & \text{on } \Gamma, \\ \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

- The pressure can be found independently of the velocity as a solution of the Dirichlet problem:

$$\Delta\, \theta = 0 \ \text{ in } \ \Omega, \qquad \theta = \theta_0 \ \text{ on } \ \Gamma$$

- We set $\boldsymbol{G} = -\nabla\theta$. Then, $\boldsymbol{u}$ and $\boldsymbol{w}$ are solutions respectively of

$$(\mathcal{E}_N^0) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

and

$$(\mathcal{E}_N^1) \quad \begin{cases} -\Delta\, \boldsymbol{w} = \boldsymbol{G} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, & \text{on } \Gamma, \\ \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

- We are reduced to solve Problem $(\mathcal{E}_N^0)$ and Problem $(\mathcal{E}_N^1)$.

Study of the elliptic problem

$$(\mathcal{E}_N^0) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. & \end{cases}$$

Remarks:

- The condition $\operatorname{div} \boldsymbol{f} = 0$ in Ω is necessary to solve (E_N^0).
- The condition $\operatorname{div} \boldsymbol{u} = 0$ in $\Omega \iff \operatorname{div} \boldsymbol{u} = 0$ on Γ on the one hand. On the other hand, since

$$\operatorname{div} \boldsymbol{u} = \operatorname{div}_\Gamma \boldsymbol{u}_{\boldsymbol{\tau}} + K \boldsymbol{u} \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \quad \text{sur } \Gamma,$$

where K denotes the mean curvature of Γ, the condition $\operatorname{div} \boldsymbol{u} = 0$ on Γ is itself equivalent, if $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, to the Fourier-Robin condition:

$$K \boldsymbol{u} \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

That means that the problem (E_N^0) is equivalent to the following:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \\ K \boldsymbol{u} \cdot \boldsymbol{n} + \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

Proposition 3.1 (Weak and Strong solutions of (E_N^0))

i) Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ satisfying the following compatibility conditions:

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and } \forall \boldsymbol{v} \in \boldsymbol{K}^2_T(\Omega), \ \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0.$$

Then, the problem (E_N^0) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisying the estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} \leq C \| \boldsymbol{f} \|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

ii) If moreover Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,6/5}(\Omega)$.

Proof. We use here only Method 1 of vector potential.

- We have $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ and

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in}\, \Omega, \quad \langle \boldsymbol{f} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I,$$

- We know that if Ω is of class $\mathcal{C}^{1,1}$, there exists a unique vector potential $\boldsymbol{\psi} \in \boldsymbol{W}^{1,6/5}(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ such that

$$\begin{aligned} \boldsymbol{f} = \textbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} &= 0 \text{ in } \Omega, \\ \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \qquad & \text{on } \Gamma, \\ \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0,\ 1 \leq j &\leq J. \end{aligned}$$

with the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,6/5}(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

- Now because $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, with

$$\operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad \boldsymbol{\psi} \cdot \boldsymbol{n} = 0, \langle \boldsymbol{\psi} \cdot \boldsymbol{n}, , 1\rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J.$$

there exists a unique vector potential $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that

$$\begin{aligned} \boldsymbol{\psi} = \mathbf{curl}\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} &= 0 \text{ in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \qquad & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I. & \end{aligned}$$

with the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

- Moreover if Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$.

Study of the elliptic problem

$$(\mathcal{E}_N^1) \quad \begin{cases} -\Delta\, \boldsymbol{w} = \boldsymbol{G} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, & \text{on } \Gamma, \\ \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I. \end{cases}$$

where

$$\boldsymbol{G} = -\nabla\theta,$$

and where $\theta \in W^{1/6,6/5}(\Omega)$ is solution of the following Dirichlet problem:

$$\Delta\,\theta = 0 \text{ in } \Omega, \qquad \theta = \theta_0 \text{ on } \Gamma.$$

with $\theta_0 \in W^{1,6/5}(\Gamma)$

Proposition 3.2 (Weak and Strong solutions of (E_N^1))

i) Let

$$\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma) \quad \text{and} \quad \theta_0 \in W^{1/6,6/5}(\Gamma)$$

satisfying the following compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^2(\Omega), \quad \int_\Gamma \theta_0 \boldsymbol{v} \cdot \boldsymbol{n} = 0.$$

Then, the problem (E_N^1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisying the estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} \leq C \Big(\| \boldsymbol{g} \times \boldsymbol{n} \|_{\boldsymbol{H}^{1/2}(\Gamma)} + \| \theta_0 \|_{W^{1/6,6/5}(\Gamma)} \Big).$$

ii) If

$$\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{H}^{3/2}(\Gamma) \quad \text{and} \quad \theta_0 \in W^{7/6,6/5}(\Gamma)$$

and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{u}$ belongs to $\boldsymbol{H}^2(\Omega)$.

Very weak solution for $(\mathcal{S}_T)$

Let $\boldsymbol{f}$, χ, g, and $\boldsymbol{h}$ with

$$\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))', \ \chi \in L^p(\Omega), \ g \in W^{-1/p,p}(\Gamma), \ \boldsymbol{h} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma),$$

with $\boldsymbol{T}^{p'}(\Omega) = \left\{ \boldsymbol{\varphi} \in \boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega); \ \mathrm{div}\, \boldsymbol{\varphi} \in W_0^{1,p'}(\Omega) \right\}$ and satisfying the compatibility conditions:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{K}_T^{p'}(\Omega), \quad \langle \boldsymbol{f}, \ \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{h} \times \boldsymbol{n}, \ \boldsymbol{\varphi} \rangle_\Gamma = 0. \tag{6}$$

$$\int_\Omega \chi \, \mathrm{d}\, \boldsymbol{x} = \langle g, \ 1 \rangle_\Gamma. \tag{7}$$

Then, the Stokes problem $(\mathcal{S}_T)$ has exactly one solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \quad &\leq \quad C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \\ &+ \quad \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)} \Big). \end{aligned} \tag{8}$$

Helmholtz Decomposition for vector fields in $\boldsymbol{L}^p(\Omega)$

For any vector field $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, we have the first following decomposition:

$$\boldsymbol{v} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{u},$$

- $\boldsymbol{z} \in \boldsymbol{K}^p_N(\Omega)$ is unique,
- $\chi \in W^{1,p}_0(\Omega)$ is unique,
- $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\boldsymbol{K}^p_T(\Omega)$, of the problem :

$$\begin{cases} -\Delta\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad (\mathbf{curl}\, \boldsymbol{u} - \boldsymbol{v}) \times \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

Helmholtz Decomposition for vector fields in $\boldsymbol{L}^p(\Omega)$

For any vector field $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, we have the second following decomposition:

$$\boldsymbol{v} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{u},$$

- $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ is unique,
- $\chi \in W^{1,p}(\Omega)$ is unique up an additive constant,
- $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\boldsymbol{K}^p_N(\Omega)$, of the problem :

$$\begin{cases} -\Delta\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0, & \text{on } \Gamma. \end{cases}$$

Question:

What happens if the previous compatibility condition is not satisfied?

Variant of the system $(\mathcal{S}_N)$:

Find $(\boldsymbol{u},\, \pi,\, \boldsymbol{c})$ such that:

$$
(\mathcal{S}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \;\text{ and }\; \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \;\text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on } \Gamma_{\mathrm{i}},\;\; 1 \le i \le I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}
$$

where $\boldsymbol{c} = (c_i)_{1 \le i \le I}$.

Theorem (Weak and Strong solutions for $(\mathcal{S}'_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that:

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,p}(\Omega)$ and constants $c_1, \ldots, c_I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}} + \|\pi_0\|_{W^{1-1/p,p}}\big),$$

and where $c_1, \ldots, c_I$ are given by

$$c_i = \langle \boldsymbol{f},\, \nabla q_i^N \rangle_\Omega - \langle \pi_0,\, \nabla q_i^N \cdot \boldsymbol{n} \rangle_\Gamma. \tag{9}$$

In particular, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

Remark :

- Observe that the following condition

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f}\,,\, \boldsymbol{v} \rangle_\Omega - \int_\Gamma \pi_0\, \boldsymbol{v} \cdot \boldsymbol{n}\ \mathrm{d}\boldsymbol{\sigma} = 0, \tag{10}$$

  is equivalent to the relations

$$c_i = 0 \quad \text{for all } i = 1, \ldots, I.$$

  Then, we have reduced to solve the problem $(\mathcal{S}'_N)$ without the constant c_i and $(\mathcal{S}'_N)$ is anything other then $(\mathcal{S}_N)$.

The assumption on $\boldsymbol{f}$ in the previous theorem can be weakened by considering the space defined for all $1 < r,\, p < \infty$:

$$\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^r(\Omega);\;\; \mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega),\;\; \boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0} \text{ on}\,\Gamma\}.$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)} = \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

We can prove that the space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)$ and its dual space can be characterized as:

$$[\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi};\;\; \boldsymbol{F} \in \boldsymbol{L}^r(\Omega),\;\; \boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)\}. \tag{11}$$

Theorem (Second Version for Weak solutions for $(\mathcal{S}'_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]', \quad \boldsymbol{g}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma),$$

with $r \leq p$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. Then, the problem $(\mathcal{S}'_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,r}(\Omega)$ and constants $c_1, \ldots, c_I$ satisfying the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &+ \|\pi\|_{W^{1,r}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \\ &+ \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big), \end{aligned}$$

and $c_1, \ldots, c_I$ are given by (9), where we replace the duality brackets on Ω by

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)}.$$

Theorem (Very weak solutions for $(\mathcal{S}_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$, and π_0 with

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]',\ \ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),\ \ \pi_0 \in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (10). Then, the Stokes problem $(\mathcal{S}_N)$ has exactly one solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{-1/p,p}(\Gamma)}\Big). \tag{12}$$

To study the case of Navier boundary conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h},$$

it suffices to observe that

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda} \boldsymbol{v} \quad \text{on}\, \Gamma,$$

where

$$\boldsymbol{\Lambda w} = \sum_{k=1}^{2} \big(\boldsymbol{w}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\big) \boldsymbol{\tau}_k.$$

VI. Oseen and Navier-Stokes Problem with Pressure Boundary Condition

We are interested to study the following problem:
Find $\boldsymbol{u}$, q and $\boldsymbol{\alpha} \in \mathbb{R}^I$ satisfying:

$$(\mathcal{NS}) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{u} \cdot \nabla\, \boldsymbol{u} + \nabla\, q = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ q = q_0 \ \text{ on } \Gamma_0 \text{ and } q = q_0 + \alpha_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases}$$

- Note that $\boldsymbol{\alpha}$ is a supplementary unknown Stokes which depends in fact on $\boldsymbol{u}$
- If we take $\chi = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, unlike the Navier-Stokes problem with Dirichlet boundary conditions de Dirichlet, the property: $\int_\Omega (\boldsymbol{u} \cdot \nabla\, \boldsymbol{u}) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} = 0$ does not hold.
- But, we have

$$\boldsymbol{u} \cdot \nabla\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \frac{1}{2}\, \nabla |\, \boldsymbol{u}|^2$$

We rewrite then $(\mathcal{NS})$ under the following form:

$$(\mathcal{NS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ sur } \Gamma_0 \text{ et } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i, \ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, & \end{cases}$$

Remarks.

- We can search directely weak solutions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ of the system $(\mathcal{NS}_N)$ by using a fixed point method.
- We can then obtain solutions $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ thanks to the Stokes problem theory.
- The case $p < 2$ to study the $(\mathcal{NS}_N)$ system is more complicated.
- For this reason, we will study the Oseen problem $(\mathcal{OS}_N)$.

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- We can search directely weak solutions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ of the system $(\mathcal{NS}_N)$ by using a fixed point method.
- We can then obtain solutions $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ thanks to the Stokes problem theory.
- The case $p < 2$ to study the $(\mathcal{NS}_N)$ system is more complicated.
- For this reason, we will study the Oseen problem $(\mathcal{OS}_N)$.

Study of problem $(\mathcal{OS}_N)$

$$(\mathcal{OS}_N) \qquad \begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \\ \pi = \pi_0 + c_i & \text{sur } \Gamma_i,\ 0 = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases} \tag{13}$$

where we have take $\chi = 0$ and $\boldsymbol{g} = \mathbf{0}$. We suppose also that

$$\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$$

We introduce the following Hilbert space:

$$\boldsymbol{V}_N = \Big\{ \boldsymbol{v} \in H^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega,\ \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \\ \text{and } \ \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0,\ \ 1 \leq i \leq I \Big\}$$

and recall that

$$\boldsymbol{v} \mapsto \Big(\int_\Omega |\mathbf{curl}\, \boldsymbol{v}|^2 \Big)^{1/2}$$

is a norm on $\boldsymbol{V}_N$ equivalent to the full norm of $\boldsymbol{H}^1(\Omega)$.

Before establishing the result of existence of a weak solution for the problem (13), we will see in what functional space it is reasonable to take π_0 and to find the pressure π appearing in (13), knowing that we are first interesting to velocity fields in $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$. With a such vector $\boldsymbol{u}$, we have $\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$. Since $\Delta\,\boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$, we deduce from the first equation in (13) that $\nabla\,\pi \in \boldsymbol{H}^{-1}(\Omega)$. Then the pressure π belongs to $L^2(\Omega)$. Furtheremore,

$$-\Delta\,\pi = \mathrm{div}\,\boldsymbol{f} - \mathrm{div}\,(\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}) \qquad \text{in } \Omega,$$

so that $\Delta\,\pi \in \boldsymbol{W}^{-1,6/5}(\Omega)$ and the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$ so that we must assume that $\pi_0 \in H^{-1/2}(\Gamma)$.

Theorem

Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$*,* $\pi_0 \in H^{-1/2}(\Gamma)$ *and* $\boldsymbol{a} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *such that* $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$*. Then, the problem:*

$$\textit{Find } (\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{V}_N \times L^2(\Omega) \times \mathbb{R}^{I+1} \textit{ satisfying } (13) \textit{ with } \langle \pi,\, 1\rangle_\Gamma = 0 \tag{14}$$

is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}_N$ *such that*

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\,\boldsymbol{x} - \langle \pi_0,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \tag{15}$$

and find constants $c_0, \ldots, c_I$ *satisfying* $\sum_{i=0}^{I} c_i \operatorname{mes} \Gamma_i + \langle \pi_0,\, 1\rangle_\Gamma = 0$ *and such that for any* $i = 1, \ldots, I$*:*

$$c_i - c_0 = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \tag{16}$$

Using the Lax Milgram theorem and some regularity result of the Laplacian, we prove the following theorem.

Theorem

Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\textbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ *and* $\pi_0 \in H^{-1/2}(\Gamma)$, *then the problem* (13) *has a unique solution* $(\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ *with* $\langle \pi, 1 \rangle_\Gamma = 0$ *and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{17}$$

$$\|\pi\|_{L^2(\Omega)} \leq C\big(1 + \|\textbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{18}$$

where $\boldsymbol{c} = (c_0, \ldots, c_I)$. *Moreover, if* $\pi_0 \in W^{1/6,6/5}(\Gamma)$ *and* Ω *is* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$ *and* $\pi \in W^{1,6/5}(\Omega)$.

Remarque

Even if the pressure π *change in* $\pi - c_0$, *the system* (13) *is equivalent to the following type-Oseen problem:*

$$(\mathcal{OS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i, \ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

where the unknowns constants satisfy for any $i = 1, \ldots, I$*:*

$$\alpha_i = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N \, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N \, \mathrm{d}\boldsymbol{x} - \langle \pi_0, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma .$$

But, it is clear that the new pressure π *does not satisfy the condition* $\langle \pi, 1 \rangle_\Gamma = 0$.

Remarque

If we suppose that $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ *and* $\pi_0 \in H^{-1/2}(\Gamma)$, *then the problems* (14) *and* (15)-(16) *are again equivalent, with the difference that we use here the duality brackets between* $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$ *and* $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)$ *in place of the integral on* Ω *in the right hand side of* (15) *and the density of* $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *in the space*

$$\boldsymbol{\mathcal{M}} = \big\{(\boldsymbol{u},\,\pi) \in \boldsymbol{H}_\sigma^1(\Omega)\times L^2(\Omega);\ \ -\Delta\,\boldsymbol{u} + \nabla\,\pi \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'\big\}.$$

It is easy now to extend Theorem 2 to the case where $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, the divergence operator does not vanish and the case of nonhomogeneous boundary conditions.

Theorem

Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$, $\chi \in W^{1,6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ *and* $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. *Then the problem*

$$\begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{19}$$

has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ *verifying the estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \quad &\leq \quad C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \\ &\times \quad \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\,\boldsymbol{g}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned}$$

$$\begin{aligned} \|\pi\|_{L^2(\Omega)} \quad &\leq C\big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \\ &+ \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\,\boldsymbol{g}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$. *Moreover, if* $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\pi_0 \in W^{1/6,6/5}(\Gamma)$, $\boldsymbol{g} \in \boldsymbol{W}^{7/6,6/5}(\Gamma)$ *and* Ω *is* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$ *and* $\pi \in W^{1,6/5}(\Omega)$.

- Strong Solutions when $p \geq 6/5$.

  In the rest of this talk, we suppose that Ω is $\mathcal{C}^{2,1}$ and we are interested in the study of strong solutions for the system $(\mathcal{OS}_N)$.
  When $p < \frac{3}{2}$, because the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p*}(\Omega)$, the term $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in L^p(\Omega)$ and we can use the regularity results on the Stokes problem. But this is not more the case when $p \geq \frac{3}{2}$ and that $\mathbf{curl}\, \boldsymbol{a}$ belongs only to $\boldsymbol{L}^{3/2}(\Omega)$.
  We give in the following theorem the good conditions to ensure the existence of strong solutions.

Theorem

Let $p \geq 6/5$,

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma), \quad \mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$$

with

$$s = \frac{3}{2} \text{ if } p < \frac{3}{2}, \qquad s = p \text{ if } p > \frac{3}{2}, \qquad s = \frac{3}{2} + \varepsilon \text{ if } p = \frac{3}{2}, \tag{20}$$

for $\varepsilon > 0$ arbitrary. Then the solution $(\boldsymbol{u}, \pi)$ given by the previous theorem belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big).$$

- Generalized Solutions with $(p > 2)$:

Theorem

Let $p > 2$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. *We suppose that* $\pi_0 \in W^{1-1/r,r}(\Gamma)$ *and* $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *and* s *satisfies:*

$$s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s = \frac{3}{2} + \varepsilon \text{ if } p = 3 \text{ and } s = r \text{ if } p > 3,$$

for some arbitrary $\varepsilon > 0$. *Then the problem* (19) *has a unique solution* $(\boldsymbol{u}, \pi, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *satisfying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \\ + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big) \quad (21)$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$.

- Generalized Solutions ($p < 2$):

Using a duality argument, we obtain the following result :

Theorem

We suppose that $p < 2$. Soit $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ and $\pi_0 \in W^{1-1/r,r}(\Gamma)$ with

$$r = 1 + \epsilon' \text{ if } p < \frac{3}{2}, \quad r = \frac{9+6\epsilon}{9+2\epsilon} \text{ if } p = \frac{3}{2} \text{ and } r = \frac{3p}{3+p} \text{ if } \frac{3}{2} < p < 2, \qquad (22)$$

$$s = (1+\epsilon')\frac{3p}{4p-3-\epsilon'(3-p)} \text{ if } p < \frac{3}{2}, \quad s = \frac{3}{2} + \epsilon \text{ if } p = \frac{3}{2} \text{ and } s = \frac{3}{2} \text{ if } \frac{3}{2} < p < 2, \qquad (23)$$

where $\epsilon,\,\epsilon' > 0$ are arbitrary. Problem $(\mathcal{OS}_N)$ has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2 (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)}),$$

$$\|\pi\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^3 (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)})$$

The Navier-Stokes problem $(\mathcal{NS}_N)$

$$(\mathcal{NS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_i \text{ and } \pi = \pi_0 + c_i & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, & \end{cases}$$

In the search of a proof of the existence of generalized solution for the Navier-Stokes equations $(\mathcal{NS}_N)$, we consider the case of small enough data.

Theorem

Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $\pi_0 \in W^{1-1/r,r}(\Gamma)$ *with* $\frac{3}{2} < p$ *and* $r = \frac{3p}{3+p}$.

i) There exists a constant $\alpha_1 > 0$ *such that, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_1,$$

then, there exists a solution $(\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *to problem* $(\mathcal{NS}_N)$ *verifying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big), \tag{24}$$

with $c_i = \langle \boldsymbol{f}, \nabla q_i \rangle_{\Omega_{r',p'}} + \int_\Gamma (\chi - \pi_0) \nabla q_i^N \cdot \boldsymbol{n} - \int_\Omega (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}) \cdot \nabla q_i^N$.

ii) Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ *such that this solution is unique, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_2.$$

For Further Reading

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