Lecture III

Inequalities for Vector Fields

Inf-Sup Conditions and Vector Potentials

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Outline

I. Basic properties of the functional spaces. Hilbertian case

Recall first definitions of the following operators, which are important in the study of several problems in fluid mechanics or in electromagnetism.

For $\boldsymbol{v} = (v_1, v_2, v_3)$, we set

$$\nabla \boldsymbol{v} = (\frac{\partial v_i}{\partial x_j})_{1 \leq i,j, \leq 3},$$

$$\text{div}\, \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \sum_{i=1}^{i=3} \frac{\partial v_i}{\partial x_i},$$

and

$$\mathbf{curl}\ \boldsymbol{v} = \nabla \times \boldsymbol{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}).$$

Even if we consider here the Hilbertian case, we define the following Banach spaces, for $1 < p < \infty$:

$$\boldsymbol{H}^{p}(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\,,\ \ \boldsymbol{H}^{p}(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\}$$

$$\boldsymbol{X}^{p}(\Omega) = \boldsymbol{H}^{p}(\mathbf{curl},\Omega) \cap \boldsymbol{H}^{p}(\mathrm{div},\Omega),$$

and their subspaces:

$$\boldsymbol{H}^{p}_{0}(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{p}(\mathbf{curl},\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}\,,$$

$$\boldsymbol{H}^{p}_{0}(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{p}(\mathrm{div},\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}\,,$$

$$\boldsymbol{X}^{p}_{N}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{p}(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ sur } \Gamma\}\,,\quad \boldsymbol{X}^{p}_{T}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{p}(\Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ sur } \Gamma\}$$

equipped with the graph norm.

Note that

$$\boldsymbol{\mathcal{D}}(\overline{\Omega}) \quad \text{is dense in} \quad \boldsymbol{H}^{p}(\mathbf{curl}, \Omega), \ \boldsymbol{H}^{p}(\mathrm{div}, \Omega) \quad \text{and} \quad \boldsymbol{X}^{p}(\Omega).$$

We have denoted by $\boldsymbol{v} \times \boldsymbol{n}$ (respectively $\boldsymbol{v} \cdot \boldsymbol{n}$) the tangential (respectively normal) boundary value of $\boldsymbol{v}$ defined in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ (respectively $W^{-\frac{1}{p},p}(\Gamma)$) as soon as $\boldsymbol{v}$ belongs to $\boldsymbol{H}^{p}(\mathbf{curl}, \Omega)$ (respectively $\boldsymbol{H}^{p}(\mathrm{div}, \Omega)$).

More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^{p}(\mathbf{curl}, \Omega)$ (respectively $\boldsymbol{H}^{p}(\mathrm{div}, \Omega)$) has a tangential (respectively normal) trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively $\boldsymbol{v} \cdot \boldsymbol{n}$) in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ (respectively $W^{-\frac{1}{p},p}(\Gamma)$), defined by

$$\langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}, \quad (1)$$

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma = \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathrm{div}\, \boldsymbol{v}) \varphi \, \mathrm{d}\boldsymbol{x}, \quad (2)$$

for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, (resp. for any $\varphi \in W^{1,p'}(\Omega)$) where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality bracket between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ in (1) and between $W^{-\frac{1}{p},p}(\Gamma)$ and $W^{\frac{1}{p},p'}(\Gamma)$ in (2).

Note also that for any $1 \le p < \infty$

$$\boldsymbol{\mathcal{D}}(\Omega) \quad \text{is dense in} \quad \boldsymbol{H}_0^p(\mathbf{curl}, \Omega) \quad \text{and in} \quad \boldsymbol{H}_0^p(\mathrm{div}, \Omega).$$

Theorem 1.1

The space $\boldsymbol{X}_0^2(\Omega) = \boldsymbol{X}_T^2(\Omega) \cap \boldsymbol{X}_N^2(\Omega)$ coincides with $\boldsymbol{H}_0^1(\Omega)$.

Proof. Since the imbedding of $\boldsymbol{H}_0^1(\Omega)$ in $X_0^2(\Omega)$ is obvious, we study the inverse imbedding.

Let $\boldsymbol{v} \in X_0^2(\Omega)$ and the extension $\tilde{\boldsymbol{v}}$ of $\boldsymbol{v}$ by $\mathbf{0}$ outside of Ω. Since $\boldsymbol{v} \in X_N(\Omega)$, it is easy to check from Green formula (1) that

$$\mathbf{curl}\, \tilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\mathbb{R}^3).$$

Similarly, the fact that $\boldsymbol{v} \in \boldsymbol{X}_T^2(\Omega)$ implies that

$$\operatorname{div} \tilde{\boldsymbol{v}} \in L^2(\mathbb{R}^3).$$

Next, thanks to Plancherel equality, the Fourier transform of $\tilde{\boldsymbol{v}}$ satisfies

$$(\xi_2 \hat{\tilde{v}}_3 - \xi_3 \hat{\tilde{v}}_2,\ \xi_3 \hat{\tilde{v}}_1 - \xi_1 \hat{\tilde{v}}_3,\ \xi_1 \hat{\tilde{v}}_2 - \xi_2 \hat{\tilde{v}}_1) \in \boldsymbol{L}^2(\mathbb{R}^3)$$

and

$$\xi_1 \hat{\tilde{v}}_1 + \xi_2 \hat{\tilde{v}}_2 + \xi_3 \hat{\tilde{v}}_3 \in L^2(\mathbb{R}^3).$$

It is then easy to check that for $1 \leq i, j, \leq 3$,

$$\|\xi_i \hat{\tilde{v}}_j\|_{L^2(\mathbb{R}^3)} \leq C(\|\mathbf{curl}\,\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} + \|\mathrm{div}\,\tilde{\boldsymbol{v}}\|_{L^2(\mathbb{R}^3)}).$$

Hence,

$$\mathbf{grad}\,\tilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\mathbb{R}^3),$$

and we obtain the theorem.

Remark. By integrating by parts and using a density argument, the following identity is readily checked for any function $\boldsymbol{v}$ in $\boldsymbol{H}_0^1(\Omega)$:

$$\|\mathbf{grad}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)}.$$

Theorem 1.2

Assume that the domain Ω is of class $\mathcal{C}^{1,1}$. Then the space $\boldsymbol{X}_T^2(\Omega)$ is continuously imbedded in $\boldsymbol{H}^1(\Omega)$

Proof.

- Step 1. We prove that

$$\boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T^2(\Omega) \quad \text{is dense in} \quad \boldsymbol{X}_T^2(\Omega).$$

Indeed, let $\boldsymbol{v} \in \boldsymbol{X}_T^2(\Omega)$. Using the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{X}^2(\Omega)$, let $\boldsymbol{v}_k \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ which converges to $\boldsymbol{v}$ in $\boldsymbol{X}^2(\Omega)$.

Next, for each k, we consider the unique solution χ_k in $H^1(\Omega)$ with zero mean value, of the problem

$$\forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla \chi_k \cdot \nabla \varphi = \int_\Omega \boldsymbol{v}_k \cdot \nabla \varphi.$$

Equivalently, it can be noted that χ_k solves the Neumann problem

$$\Delta \chi_k = \operatorname{div} \boldsymbol{v}_k \quad \text{in } \Omega \quad \text{and} \quad \partial_n \chi_k = \boldsymbol{v}_k \cdot \boldsymbol{n} \text{ on } \Gamma.$$

Due to the regularity assumption on the domain Ω, for each k, the function χ_k belongs to $H^2(\Omega)$, so that the vector field $\boldsymbol{v}_k - \nabla\chi_k$ is in $\boldsymbol{H}^1(\Omega)$.

Finally, due to the convergence of $(\boldsymbol{v}_k)_k$ in $\boldsymbol{X}^2(\Omega)$, it is easy to check that the sequence $(\nabla\chi_k)_k$ converges in $\boldsymbol{L}^2(\Omega)$ towards $\nabla\chi$, with $\chi \in H^2(\Omega)$ solution of the problem

$$\Delta\chi = \operatorname{div} \boldsymbol{v} \quad \text{in } \Omega \quad \text{and} \quad \partial_n\chi = \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,$$

and where we observe that

$$\int_\Omega |\nabla(\chi_k - \chi)|^2 = \int_\Omega (\boldsymbol{v}_k - \boldsymbol{v})\cdot\nabla(\chi_k - \chi).$$

Hence, the sequence $(\boldsymbol{v}_k - \nabla\chi_k + \nabla\chi)_k$ is in $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T^2(\Omega)$ and converges to $\boldsymbol{v}$ in $\boldsymbol{X}_T^2(\Omega)$, which proves the density.

- Step 2. We use the following inequality:

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T^2(\Omega)$

$$\begin{aligned} \|\mathbf{grad}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} &= \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)} \qquad (3) \\ &- \int_\Gamma \mathcal{B}(\boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{v} \times \boldsymbol{n})\, \mathrm{d}\boldsymbol{\tau}, \qquad (4) \end{aligned}$$

where $\mathcal{B}$ denotes the curvature tensor of the boundary.

But

$$\begin{aligned} \left|\int_\Gamma \mathcal{B}(\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{v}\times\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau}\right| &\leq C_1\int_\Gamma |\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{\tau} \\ &\leq \frac{1}{2}\|\mathbf{grad}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + C_2\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

We deduce then the following inequality:

$$\text{for any } \boldsymbol{v}\in\boldsymbol{H}^1(\Omega)\cap\boldsymbol{X}^2_T(\Omega),$$

$$\|\mathbf{grad}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)}).$$

- Finally, we prove the theorem.

Theorem 1.3

Assume that the domain Ω is of class $\mathcal{C}^{1,1}$. Then the space $\boldsymbol{X}_N^2(\Omega)$ is continuously imbedded in $\boldsymbol{H}^1(\Omega)$

Proof.

- Firstly we prove that

$$\boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N^2(\Omega) \quad \text{is dense in} \quad \boldsymbol{X}_N^2(\Omega)$$

by using the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{X}^2(\Omega)$ and by solving the following problem: Find $\boldsymbol{\xi} \in \boldsymbol{H}^2(\Omega)$ such that

$$\begin{aligned} \boldsymbol{\xi} - \Delta\boldsymbol{\xi} = \mathbf{curl}\, \boldsymbol{v}, \quad \operatorname{div} \boldsymbol{\xi} = 0 \qquad &\text{in} \quad \Omega, \\ \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \quad \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0} \quad &\text{on} \quad \Gamma, \end{aligned}$$

with $\boldsymbol{v}$ belonging to $\boldsymbol{X}_N^2(\Omega)$.

- Secondly, we use the following inequality: for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N^2(\Omega)$

$$\begin{aligned} \|\mathbf{grad}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} &= \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)} \quad (5) \\ &- \int_\Gamma (\mathrm{Tr}\, \mathcal{B})(\boldsymbol{v} \cdot \boldsymbol{n})^2 \, \mathrm{d}\boldsymbol{\tau}. \quad (6) \end{aligned}$$

Like previously we deduce the following inequality:

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N^2(\Omega)$

$$\|\mathbf{grad}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)}).$$

- Finally, we prove the theorem.

More generally, setting

$$\begin{aligned} \boldsymbol{X}^{m,p}(\Omega) &= \{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v}\in W^{m-1,p}(\Omega), \\ &\quad \mathbf{curl}\,\boldsymbol{v}\in \boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n}\in W^{m-\frac{1}{p},p}(\Gamma)\}, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{Y}^{m,p}(\Omega) &= \{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v}\in W^{m-1,p}(\Omega), \\ &\quad \mathbf{curl}\,\boldsymbol{v}\in \boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\times\boldsymbol{n}\in \boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)\}, \end{aligned}$$

then we have the following regularity result

Theorem 1.4

Let $m\in\mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then $\boldsymbol{X}^{m,2}(\Omega)$ is continuously imbedded in $\boldsymbol{H}^m(\Omega)$ and for any $\boldsymbol{v}$ in $\boldsymbol{H}^m(\Omega)$:

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{H}^m(\Omega)} &\le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{H}^{m-1}(\Omega)} \\ &+ \|\operatorname{div}\boldsymbol{v}\|_{H^{m-1}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{m-\frac{1}{2}}(\Gamma)}\big), \end{aligned}$$

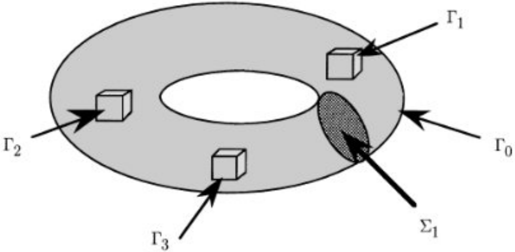
with similar properties for the space $\boldsymbol{Y}^{m,2}(\Omega)$.

II. L^2-Theory for Vector Potentials

We suppose that Ω is an open set possibly multiply connected sufficiently regular with a boundary Γ non connected. We denote $\Gamma = \bigcup_{i=0}^{I} \Gamma_i$ with Γ_i the connected components of Γ

and $\Sigma = \bigcup_{j=1}^{J} \Sigma_j$ and Σ_j a finite number of cuts.

$\Omega^\circ = \Omega \setminus \Sigma$ is simply connected.



Theorem 2.1 (General vector potentials)

A vector field $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ satisfies:

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\,\Omega, \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I, \tag{7}$$

iff there exists a vector potential $\boldsymbol{\psi}_0$ dans $\boldsymbol{H}^1(\Omega)$ such that

$$\boldsymbol{u} = \textbf{curl}\,\boldsymbol{\psi}_0 \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi}_0 = 0 \text{ in } \Omega, \tag{8}$$

Remark: Note that the condition $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ makes sense because, thanks to Green Formula (2), the restriction of $\boldsymbol{u}\cdot\boldsymbol{n}$ to each Γ_i belongs to $H^{-1/2}(\Gamma_i)$.

Proof: i) Let $\boldsymbol{u}$ be any function satisfying (7). Using the above notation, for $0 \le i \le I$, we consider the solution $\chi_i \in H^1(\Omega_i)$ of the following Neumann problem

$$\begin{aligned} \Delta\chi_0 &= 0 \quad \text{in } \Omega_0, \quad \frac{\partial \chi_0}{\partial \boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n} \quad \text{on } \Gamma_0 \quad \text{and} \quad \frac{\partial \chi_0}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\mathcal{O} \\ \Delta\chi_i &= 0 \quad \text{in } \Omega_i, \quad \frac{\partial \chi_i}{\partial \boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n} \quad \text{on } \Gamma_i. \end{aligned}$$

Then the function $\tilde{\boldsymbol{u}}$ defined by

$$\tilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{grad}\,\chi_i & \text{in } \Omega_i,\ 0 \le i \le I \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \partial\mathcal{O} \end{cases}$$

belongs to $\boldsymbol{L}^2(\mathbb{R}^3)$ with divergence-free in $\mathbb{R}^3$.

Taking its Fourier transform and denoting it simply by $\hat{\boldsymbol{u}}$ leads to the equation

$$\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3 = 0.$$

Next, observe that conditions (8) are satisfied by a function $\boldsymbol{\psi}_0$ if and only if

$$\hat{u}_1 = \xi_2 \hat{\psi}_{03} - \xi_3 \hat{\psi}_{02},\ \hat{u}_2 = \xi_3 \hat{\psi}_{01} - \xi_1 \hat{\psi}_{03},\ \hat{u}_3 = \xi_1 \hat{\psi}_{02} - \xi_2 \hat{\psi}_{01}, \quad (9)$$

and

$$\xi_1 \hat{\psi}_{01} + \xi_2 \hat{\psi}_{02} + \xi_3 \hat{\psi}_{03} = 0. \quad (10)$$

In $L^2(\mathbb{R}^3)$, system (9)- (10) is equivalent to

$$\hat{\psi}_{01} = \frac{\xi_2 \hat{u}_3 - \xi_3 \hat{u}_2}{|\boldsymbol{\xi}|^2},\ \hat{\psi}_{02} = \frac{\xi_1 \hat{u}_3 - \xi_3 \hat{u}_1}{|\boldsymbol{\xi}|^2},\ \hat{\psi}_{03} = \frac{\xi_2 \hat{u}_1 - \xi_1 \hat{u}_2}{|\boldsymbol{\xi}|^2} \quad (11)$$

Let us define the function $\boldsymbol{\psi}_0$ by equations (11). Its gradient is clearly in $\boldsymbol{L}^2(\Omega)$ due to the inequalities

$$|\xi_j \hat{\psi}_{0k}| \leq \sum_{\ell=1}^{3} |\hat{u}_\ell|$$

ii) Conversely, for any $\boldsymbol{\psi}_0 \in \boldsymbol{H}^1(\Omega)$,

$$\operatorname{div}(\mathbf{curl}\,\boldsymbol{\psi}_0) = 0.$$

Moreover, for $0 \leq i \leq I$, let ν_i be a function of class $\mathcal{C}^\infty$ on $\overline{\Omega}$ which is equal to 1 in a neighbourhood of Γ_i and vanishes in a neighbourhood of Γ_k, with $0 \leq k \leq I$, $k \neq i$. We have

$$\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = \langle \mathbf{curl}\,(\boldsymbol{\nu_i}\boldsymbol{\psi}_0) \cdot \boldsymbol{n},\, 1\rangle_{\Gamma} = \int_\Omega \operatorname{div}(\mathbf{curl}\,(\boldsymbol{\nu_i}\boldsymbol{\psi}_0))d\boldsymbol{x} = 0$$

which is the desired condition.

We require now some preliminaries.

First, for any function q in $H^1(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distributions in $\mathcal{D}'(\Omega^\circ)$. It belongs to $\boldsymbol{L}^2(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^2(\Omega)$. In order to distinguish this extension from the gradient of q in $\mathcal{D}'(\Omega)$, we denote it by

$$\widetilde{\mathbf{grad}}\, q.$$

Lemma 2.2 (Green Formula)

If $\boldsymbol{\psi}$ belongs to $\boldsymbol{H}_0^2(\mathrm{div}, \Omega)$, the restriction of $\boldsymbol{\psi} \cdot \boldsymbol{n}$ to any Σ_j belongs to the dual space $[\boldsymbol{H}_{00}^{1/2}(\Sigma_j)]'$, and the following Green's formula holds: $\forall \chi \in H^1(\Omega^\circ)$,

$$\sum_{j=1}^{J} \langle \boldsymbol{\psi} \cdot \boldsymbol{n}, [\,\chi\,]_j \rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi} \cdot \mathbf{grad}\, \chi \, d\boldsymbol{x} + \int_{\Omega^\circ} \chi \, \mathrm{div}\, \boldsymbol{\psi} \, d\boldsymbol{x}, \quad (12)$$

where we recall that $[\chi]_j$ is the jump of χ through Σ_j.

We introduce the following space

$$\Theta^2(\Omega^\circ) = \left\{ r \in H^1(\Omega^\circ);\ [\, r\,]_j = \text{constant},\ 1 \le j \le J \right\}.$$

Using the previous Green formula, it is easy to prove the following lemma.

Lemma 2.3 (Characterization of $\Theta^2(\Omega^\circ)$)

Let r belong to $H^1(\Omega^\circ)$. Then r belongs to $\Theta^2(\Omega^\circ)$ if and only if

$$\mathbf{curl}(\widetilde{\mathbf{grad}}\, r) = \mathbf{0} \quad \text{in}\, \Omega.$$

Theorem 2.4 (Tangent Vector Potential)

A vector field $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ satisfies:

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\, \Omega, \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I,$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{H}^1(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \\ &\boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \quad \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_\mathrm{j}} = 0,\ 1 \leq \mathrm{j} \leq \mathrm{J}. \end{aligned} \tag{13}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}. \tag{14}$$

Before to prove this theorem, we will give some preliminary results.

Remark. **i)** The statement of this theorem is independent of the particular choice of the admissible set of cuts $\{\Sigma_j;\ 1 \leq j \leq J\}$.

ii) Clearly the uniqueness of the function $\boldsymbol{\psi}$ will follow from the characterization of the kernel

$$\begin{aligned} \boldsymbol{K}^2_T(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div} \boldsymbol{v} = 0, \operatorname{\mathbf{curl}} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega \\ &\qquad \text{and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\} \end{aligned}$$

Proposition 2.5 (Characterization of $\boldsymbol{K}^2_T(\Omega)$)

The dimension of the space $\boldsymbol{K}^2_T(\Omega)$ is equal to J. It is spanned by the functions $\widetilde{\textbf{grad}}\, q^T_j$, $1 \le j \le J$, where each $q^T_j \in H^1(\Omega^\circ)$ is the solution, unique up to an additive constant, of the problem

$$\begin{cases} -\Delta q^T_j = 0 \qquad \text{in } \Omega^\circ, \\ \partial_n\, q^T_j = 0 \qquad \text{on } \Gamma, \\ \left[\, q^T_j \,\right]_k = \text{constant} \quad \text{and} \quad [\, \partial_n\, q^T_j \,]_k = 0,\ 1 \le k \le J, \\ \left\langle \partial_n\, q^T_j ,\, 1 \right\rangle_{\Sigma_k} = \delta_{j\,k},\ 1 \le k \le J, \end{cases} \tag{15}$$

Proof. This problem is in fact equivalent to the problem:

Find $q_j^T \in \Theta^2(\Omega^\circ)$ such that

$$\forall r \in \Theta^2(\Omega^\circ), \quad \int_{\Omega^\circ} \mathbf{grad}\, q_j^T \cdot \mathbf{grad}\, r\, d\boldsymbol{x} = [\, r\,]_j$$

which has a solution, unique up to an additive constant, by using Lax-Milgram Lemma. Note that Ω° is not a Lipschitzian domain.

Note also that for any $r \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \left\langle \operatorname{div}(\widetilde{\mathbf{grad}}\, q_j^T),\, r \right\rangle &= -\int_{\Omega} \widetilde{\mathbf{grad}}\, q_j^T \cdot \mathbf{grad}\, r\, d\boldsymbol{x} \\ &= -\int_{\Omega^\circ} \mathbf{grad}\, q_j^T \cdot \mathbf{grad}\, r\, d\boldsymbol{x} = 0, \end{aligned}$$

As an immediate consequence of this proposition, the compactness of $\boldsymbol{X}_T^2(\Omega)$ into $\boldsymbol{L}^2(\Omega)$ and Peetre-Tartar Theorem, we have

Corollary 2.6 (Equivalence of Norms)

On the space $\boldsymbol{X}_T^2(\Omega)$, the seminorm

$$\boldsymbol{v} \mapsto \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \sum_{\mathrm{j}=1}^{\mathrm{J}} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}}|, \qquad (16)$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^2(\Omega)}$. In particular, we have the following inequality for every function $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\Big(\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{j=1}^{J} \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\Big). \qquad (17)$$

Proof of Theorem "Tangent Vector Potential"
Existence

1. Assume that (7) and let $\boldsymbol{\psi}_0 \in \boldsymbol{H}^1(\Omega)$ denote the general potential vector associated with $\boldsymbol{u}$:

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0 \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi}_0 = 0 \ \text{ in } \Omega.$$

2. Let $\chi \in H^1(\Omega)$ be the solution of the problem :

$$-\Delta\,\chi = 0 \ \text{ in } \ \Omega \qquad \text{and} \quad \partial_n\,\chi = \boldsymbol{\psi}_0 \cdot \boldsymbol{n} \ \text{ on}\,\Gamma.$$

3. Setting $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \mathbf{grad}\,\chi$, then the function

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \sum_{j=1}^{J} \langle\, \boldsymbol{\psi}_1 \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j}\ \widetilde{\mathbf{grad}}\, q_j^T$$

is the vector potential required.

Uniqueness : The uniqueness of this function $\boldsymbol{\psi}$ is a consequence of the characterization of the kernel $\boldsymbol{K}_T^2(\Omega)$. Note that $\boldsymbol{K}_T^2(\Omega) = \{0\}$ if Ω is simply connected.

Theorem 2.7 (Normal vector potentials)

A vector field $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ satisfies:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \quad 0 \leq j \leq J,$$

iff there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{X}^2(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \tag{18}$$

$$\boldsymbol{\psi} \times \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{19}$$

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I. \tag{20}$$

This function is unique and moreover, we have:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

Remark : If $\boldsymbol{u} \in \boldsymbol{H}_0^2(\mathrm{div},\, \Omega)$ then the condition $\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J$, is necessary and sufficient for the existence of the vector potential $\boldsymbol{\psi}$ satisfying (69) and (69). The condition $\langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0,\ 0 \leq i \leq I$ ensures the uniqueness of $\boldsymbol{\psi}$.

As previously, the uniqueness result is linked to the characterization of the kernel

$$\boldsymbol{K}^2_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \ \text{ in } \Omega \text{ and } \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma$$

Proposition 2.8 (Normal Vector Potential)

The dimension of the space $\boldsymbol{K}^2_N(\Omega)$ is equall to I. It is spanned by the functions $\mathbf{grad}\, q_j^N,\ 1 \le j \le J$, where each $q_j^N \in H^1(\Omega)$ is the unique solution of the problem

$$(\mathcal{P}_N) \begin{cases} -\Delta q_i^N = 0 \quad \text{in } \Omega, \quad q_i^N = 0 \ \text{ on } \Gamma_0, \ \ q_i^N = \text{constant} \ \text{ on } \Gamma_k, \\ \left\langle \partial_n\, q_i^N,\, 1 \right\rangle_{\Gamma_0} = -1 \ \text{ and } \ \left\langle \partial_n\, q_i^N,\, 1 \right\rangle_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I. \end{cases}$$

Proof. Let $\Theta(\Omega)$ denote the space

$$\Theta(\Omega) = \left\{ r \in H^1(\Omega);\ r|_{\Gamma_0} = 0 \quad \text{and}\, r|_{\Gamma_i} = \text{constant},\ 1 \le i \le I \right\}.$$

This problem is in fact equivalent to the problem:

Find $q_i^N \in \Theta(\Omega)$ such that

$$\forall r \in \Theta(\Omega), \quad \int_\Omega \mathbf{grad}\, q_i^N \cdot \mathbf{grad}\, r\, d\boldsymbol{x} = r|_{\Gamma_i},$$

which has a unique solution, by using Lax-Milgram Lemma.

The functions $\mathbf{grad}\, q_j^N$ for $1 \le i \le I$ are obviously independent and belong to $\boldsymbol{K}_N^2(\Omega)$. It remains to prove that they span $\boldsymbol{K}_N^2(\Omega)$. Take any function $\boldsymbol{w}$ in $\boldsymbol{K}_N^2(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{i=1}^{I} \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} \mathbf{grad}\, q_i^N.$$

It is easy to prove that $\boldsymbol{u}$ satisfies (7), so that it can be written $\mathbf{curl}\,\boldsymbol{\psi}_0$, for some $\boldsymbol{\psi}_0$ in $\boldsymbol{H}^1(\Omega)$. This allows to compute

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u} = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\psi}_0 = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{\psi}_0 + \langle \boldsymbol{u}\times\boldsymbol{n},\, \boldsymbol{\psi}_0\rangle_\Gamma = 0$$

so that $\boldsymbol{u}$ is equal to $\mathbf{0}$. That ends the proof.

As previously this proposition has a corollary about equivalent norms.

Corollary 2.9 (Equivalence of Norms)

On the space $\boldsymbol{X}^2_N(\Omega)$, the seminorm

$$\boldsymbol{v} \mapsto \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_\mathrm{i}}|, \tag{21}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^2(\Omega)}$. In particular, we have the following inequality for every function $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\Big(\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}\Big). \tag{22}$$

Proof of Theorem "Normal Vector Potential" The proof is divided into three steps.

- **Step 1. Necessary conditions**. We assume that

$$\begin{aligned} \boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } \Omega, \\ \boldsymbol{\psi}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \\ \langle \boldsymbol{\psi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 1 \le i \le I. \end{aligned}$$

It is clear that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ is divergence-free. Moreover for any $\chi \in H^2(\Omega)$, Green formulas yield

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi = \langle \boldsymbol{u}\cdot\boldsymbol{n},\, \chi\rangle_\Gamma\,, \tag{23}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi = -\langle \boldsymbol{\psi}\times\boldsymbol{n},\, \mathbf{grad}\,\chi\rangle_\Gamma\,. \tag{24}$$

Therefore if $\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0}$ on Γ, a density argument gives $\mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n} = 0$ on Γ. Hence,

$$\mathbf{curl}\,\boldsymbol{\psi} \in \mathbf{H}_0^2(\operatorname{div},\, \Omega)$$

and by Green Formula, we prove that

$$\langle \mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0 \quad \text{for} \quad 1 \le j \le J.$$

- **Step 2. Existence of the normal potential vector.**
We know that there exists $\boldsymbol{\psi}_0 \in \boldsymbol{H}^1(\Omega)$ with

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0 \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi}_0 = 0.$$

Setting

$$\boldsymbol{V}^2_T(\Omega) \quad = \quad \{\boldsymbol{v} \in \boldsymbol{X}^2_T(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ \ 1 \le j \le J\},$$

Then using Lax-Milgram Lemma, the following problem:
Find $\boldsymbol{\xi}$ in $\boldsymbol{V}^2_T(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{V}^2_T(\Omega)$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \tag{25}$$

has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}^2_T(\Omega)$. Note that the right-hand side defines an element of $(\boldsymbol{V}^2_T(\Omega))'$.
Next, we want to extend this formulation to any test function in $\boldsymbol{X}^2_T(\Omega)$.

For that, let $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^2(\Omega)$ and χ in $H^2(\Omega)$ satisfying:

$$\Delta\,\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}}\ \text{ in}\,\Omega \quad \text{and} \quad \frac{\partial\,\chi}{\partial \boldsymbol{n}} = 0\ \text{ on}\,\Gamma. \tag{26}$$

Let then $\boldsymbol{\varphi} \in \boldsymbol{V}_T^2(\Omega)$ satisfying:

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle(\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi)\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T. \tag{27}$$

Observe that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} == 0,$$

and we obtain

$$\begin{aligned}
\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \widetilde{\mathbf{grad}}\, q_j^T\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega^\circ} \boldsymbol{u}\cdot\mathbf{grad}\,q_j^T\,\mathrm{d}\boldsymbol{x} \\
&= \sum_{k=1}^{J} [q_j^T]_k \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_k} + \langle \boldsymbol{u}\cdot\boldsymbol{n},\, q_j^T\rangle_\Gamma = 0.
\end{aligned}$$

Hence, (25) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^2(\Omega)$ such that for any $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^2(\Omega)$:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}. \quad (28)$$

In fact, every solution of (28) also solves the problem

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{0}, \quad \operatorname{div}\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0,\ (\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi})\times\boldsymbol{n} = \mathbf{0} & \text{on}\,\Gamma, \\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ 1\le j\le J. & \end{cases}$$

Finally, setting $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi}$, and

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \sum_{i=1}^{I} \langle \boldsymbol{\psi}_1 \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N,$$

it follows that the function $\boldsymbol{\psi}$ belongs to $\boldsymbol{L}^2(\Omega)$ and satisfies the required properties. Observe that $\boldsymbol{\xi} \in \boldsymbol{H}^2(\Omega)$ and then $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$.

- **Step 3. Uniqueness.** The uniqueness of this function $\boldsymbol{\psi}$ is a consequence of the characterization of the kernel $\boldsymbol{K}_N^2(\Omega)$.

Note that $\boldsymbol{K}_N^2(\Omega) = \{0\}$ if Γ is connected

III. Inequalities for Vector Fields. General L^p-theory

Theorem 3.1 (Sobolev's inequalities I)

Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega)$ satisfies:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\Big). \qquad (29)$$

W. Von Wahl (1992) ($I = 0$, *i.e* Γ is connected).

Proof.

- We introduce the linear integral operator:

$$T\,\lambda(\boldsymbol{x}) = -\frac{1}{2\pi}\int_{\Gamma}\lambda(\boldsymbol{\xi})\,\frac{\partial}{\partial \boldsymbol{n}}|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\,\mathrm{d}\sigma_{\boldsymbol{\xi}},$$

  $T:\ L^p(\Gamma) \longrightarrow W^{1,p}(\Gamma)$ is continuous and consequently compact from $L^p(\Gamma)$ into $L^p(\Gamma)$. By Fredholm alternative the space Ker$(Id+T)$ is of finite dimension (equal to I) and Im(Id +T) is closed. Then the operator $Id+T$ is linear, continuous and surjective from $L^p(\Gamma)$ onto Im(Id +T). Using the theorem of open map, for any $\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega)$ we have :

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\big(\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{i=1}^{I}|\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}|\big) \qquad (30)$$

- For any $\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, we have the following intégral représentation:

$$\begin{aligned}(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n}) &= -\frac{1}{2\pi}\left(\mathbf{grad}\int_{\Omega}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\\ &- \frac{1}{2\pi}\left(\mathbf{curl}\int_{\Omega}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n} \qquad (31)\end{aligned}$$

- Using the trace inequality, we prove:

$$\begin{aligned}\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{\boldsymbol{L}^p(\Gamma)} &\leq C\Big(\Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)} \\ &+ \Big\|\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

- Using then the Calderón-Zygmund inequalities, we get:

$$\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} \leq C\left(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\right).$$

- From (30) we obtain directly:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\Big). \quad (32)$$

- Moreover, from (31) and using triangular inequality we have:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq \Big(\|T(\boldsymbol{v}\cdot\boldsymbol{n})\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}$$
$$+ \Big\|\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big),$$

and thanks to the trace's theorem we have:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\Big(\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} + \Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}$$
$$+ \Big\|\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).$$

- Using again the Calderón-Zygmund inequalities and (32), we get:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\big(\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\big). \quad (33)$$

- As $\boldsymbol{v} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, by the trace's theorem, there exists $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\boldsymbol{v} = \boldsymbol{u} \quad \text{on}\, \Gamma \qquad \text{and} \quad \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{v}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Because $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ, $\boldsymbol{v}|_\Gamma = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$, using then (33) we get successively

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\Big). \quad (34)$$

- Because $\boldsymbol{u} - \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we know that for any function $\boldsymbol{w}$ in $\boldsymbol{W}_0^{1,p}(\Omega)$, we have the following integral representation:

$$\boldsymbol{w} = -\mathbf{grad}\, \frac{1}{4\pi} \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathrm{div}_{\boldsymbol{y}}\, \boldsymbol{w}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} + \mathbf{curl}\, \frac{1}{4\pi} \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathbf{curl}_{\boldsymbol{y}}\, \boldsymbol{w}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.$$

Using again the Calderón-Zygmund inequalities, we get

$$\|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\mathrm{div}\, \boldsymbol{w}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big). \quad (35)$$

- Applying (35) to $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we obtain:

$$\|\nabla (\boldsymbol{v} - \boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\Big).$$

Finally, we deduce the required estimate by using directly (34).

Using the Hahn-Banach Theorem, we prove the following lemma

Lemma 3.2

The space $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega)$ is dense in the space $\boldsymbol{X}_N^p(\Omega)$.

Proof. Let $\boldsymbol{\ell}$ belongs to $(\boldsymbol{X}_N^p(\Omega))'$, the dual space of $\boldsymbol{X}_N^p(\Omega)$. We know that there exist $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Omega)$ and $h \in L^{p'}(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega)$,

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \, \mathrm{div}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \qquad (36)$$

We suppose that

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega), \qquad \langle \boldsymbol{\ell}, \, \boldsymbol{v} \rangle = 0. \qquad (37)$$

So, we have in the sense of distributions in Ω

$$\boldsymbol{f} - \nabla h + \mathbf{curl}\, \boldsymbol{g} = \mathbf{0}. \tag{38}$$

Therefore, due to (37) and (36), we have :
for any $\chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\int_\Omega \boldsymbol{f} \cdot \nabla\chi \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \, \Delta\chi \, \mathrm{d}\boldsymbol{x} = 0. \tag{39}$$

Note that $\operatorname{div} \boldsymbol{f} = \Delta h \in W^{1,p'}(\Omega)$. Because $h \in L^{p'}(\Omega)$, we know that $h_{|_\Gamma} \in W^{-1/p',p'}(\Gamma)$ and we have:
for any $\chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\int_\Omega h \, \Delta \, \chi \, \mathrm{d}\boldsymbol{x} - \langle \operatorname{div} \boldsymbol{f}, \, \chi \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \langle \, h, \, \frac{\partial \, \chi}{\partial \boldsymbol{n}} \, \rangle_\Gamma.$$

As

$$\int_\Omega \boldsymbol{f}\cdot\nabla\chi\,\mathrm{d}\boldsymbol{x} = -\langle \operatorname{div}\boldsymbol{f},\,\chi\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)},$$

it follows from (39) that

$$\langle\, h,\, \frac{\partial\,\chi}{\partial\boldsymbol{n}}\,\rangle_\Gamma = 0, \qquad \forall\chi\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega). \tag{40}$$

Now, let μ be any element of $W^{1-\frac{1}{p},p}(\Gamma)$. Then, there exists an element χ of $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\frac{\partial\,\chi}{\partial\boldsymbol{n}} = \mu$ on Γ. Hence, (40) implies that

$$\langle\, h,\, \mu\,\rangle_{W^{-\frac{1}{p'},p'}(\Gamma)\times W^{1-\frac{1}{p},p}(\Gamma)} = 0,$$

and $h = 0$ in $W^{-\frac{1}{p'},p'}(\Gamma)$. Because $\Delta\,h$ belongs to $W^{-1,p'}(\Omega)$ and $h\in L^{p'}(\Omega)$, then $h\in W_0^{1,p'}(\Omega)$. As a consequence, due to (38), $\mathbf{curl}\,\boldsymbol{g}$ belongs to $\boldsymbol{L}^{p'}(\Omega)$. Finally, let $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$.

From (38) and since $h \in W_0^{1,p}(\Omega)$, we can write

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega). \tag{41}$$

As $\boldsymbol{g} \in \boldsymbol{H}^{p'}(\mathbf{curl}, \Omega)$, we have also

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathbf{curl}, \Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

Then it follows from the last equality and (41) that $\boldsymbol{\ell}$ vanishes on $\boldsymbol{X}_N^p(\Omega)$, thus proving the required density.

As a consequence, we have the following result:

Theorem 3.3 (Imbedding of $\boldsymbol{X}^{p}_{N}(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$)

The space $\boldsymbol{X}^{p}_{N}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant C, such that for any $\boldsymbol{v}$ in $\boldsymbol{X}^{p}_{N}(\Omega)$:

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \; & C \; \Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^{p}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^{p}(\Omega)} + \\ & + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i}| \Big). \end{aligned} \tag{42}$$

As an immediate consequence of this theorem, the compactness of $\boldsymbol{X}_N^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ and Peetre Theorem, we have

Corollary 3.4 (Equivalence of Norms)

On the space $\boldsymbol{X}_N^p(\Omega)$, the seminorm

$$\boldsymbol{v} \mapsto \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{\mathrm{L^p(\Omega)}} + \sum_{\mathrm{i=1}}^{\mathrm{I}} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_{\mathrm{i}}}|, \tag{43}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. In particular, we have the following inequality for every function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}\Big). \tag{44}$$

Theorem 3.5 (Sobolev's inequalities II)

Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}^{p}_{T}(\Omega)$ satisfies:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}|\Big). \qquad (45)$$

W. Von Wahl (1992) ($J = 0$, $i.e$ Ω is simply connected).

Idea of the proof of (45):

- We introduce the linear integral operator

$$R\boldsymbol{\lambda}(\boldsymbol{x}) = \frac{1}{2\pi}\int_\Gamma \mathbf{curl}\,\Big(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\Big) \times \boldsymbol{n}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}.$$

$R:\ \boldsymbol{L}^p(\Gamma) \longrightarrow \boldsymbol{W}^{1,p}(\Gamma)$ is continuous and then compact from $\boldsymbol{L}^p(\Gamma)$ into $\boldsymbol{L}^p(\Gamma)$. Fredholm alternative implies that $\mathrm{Ker}(Id+R)$ is of finite dimension (equal to J) and for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$:

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{L^p(\Gamma)} \le C\Big(\|(Id+R)(\boldsymbol{v}\times\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{j=1}^{J}|\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}|\Big) \qquad (46)$$

- For any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ, we have :

$$\begin{aligned}(Id+R)(\boldsymbol{v}\times\boldsymbol{n}) &= \frac{1}{2\pi}\Big(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big)\times\boldsymbol{n} \\ &+ \frac{1}{2\pi}\Big(\mathbf{grad}\int_\Gamma \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\xi})\,\mathrm{d}\sigma_{\boldsymbol{\xi}}\Big)\times\boldsymbol{n} \qquad (47) \\ &- \frac{1}{2\pi}\Big(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big)\times\boldsymbol{n}.\end{aligned}$$

and the rest of the proof is similar to the case of the operator T.

As for the case $p = 2$, we prove the following lemma

Lemma 3.6

The space $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^p(\Omega)$ is dense in the space $\boldsymbol{X}_T^p(\Omega)$.

As a consequence, we have the following result:

Theorem 3.7 (Imbedding of $\boldsymbol{X}_T^p(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$)

The space $\boldsymbol{X}_T^p(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant C, such that for any $\boldsymbol{v}$ in $\boldsymbol{X}_T^p(\Omega)$:

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \; & C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \\ & \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_i}|\Big). \end{aligned} \tag{48}$$

As an immediate consequence of this theorem, the compactness of $\boldsymbol{X}_T^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ and Peetre Theorem, we have

Corollary 3.8 (Equivalence of Norms)

On the space $\boldsymbol{X}_T^p(\Omega)$, the seminorm

$$\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{\mathrm{L^p(\Omega)}} + \sum_{\mathrm{j=1}}^{\mathrm{J}} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}}|, \tag{49}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. In particular, we have the following inequality for every function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \sum_{j=1}^{J}\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\Big). \tag{50}$$

Recall the following spaces

$$\begin{aligned}\boldsymbol{X}^{m,p}(\Omega) &= \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v}\in W^{m-1,p}(\Omega),\\ &\qquad \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n}\in W^{m-\frac{1}{p},p}(\Gamma)\},\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Y}^{m,p}(\Omega) &= \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v}\in W^{m-1,p}(\Omega),\\ &\qquad \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)\},\end{aligned}$$

then we have the following regularity result

Theorem 3.9

Let $m\in\mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then $\boldsymbol{X}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and for any $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$:

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} &\le C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)}\\ &+ \|\operatorname{div}\boldsymbol{v}\|_{W^{m-1,p}(\Omega)}+\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{m-\frac{1}{p},p}(\Gamma)}\Big),\end{aligned}$$

with similar properties for the space $\boldsymbol{Y}^{m,p}(\Omega)$.

IV. L^p-Theory for Vector Potentials

To study the Stokes problems, we need some results concerning the vector potentials and the inf-sup conditions.

Theorem 4.1 (General vector potentials)

$\boldsymbol{u} \in \boldsymbol{H}^{p}(\mathrm{div}, \Omega)$ satisfies:

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in}\, \Omega, \quad \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \tag{51}$$

iff there exists a vector potential $\boldsymbol{\psi}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}_0 \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\psi}_0 = 0 \text{ in } \Omega, \tag{52}$$

Proof: Here, we will construct only the general vector potential $\boldsymbol{\psi}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$. Let $\boldsymbol{u}$ be any function satsfying (51). As for the case $p = 2$, we consider for $0 \leq i \leq I$ the solution $\chi_i \in W^{1,p}(\Omega_i)$ of the following Neumann problem

$$\begin{aligned} \Delta\chi_0 &= 0 \quad \text{in } \Omega_0, \quad \frac{\partial \chi_0}{\partial \boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n} \quad \text{on } \Gamma_0 \quad \text{and} \quad \frac{\partial \chi_0}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\mathcal{O} \\ \Delta\chi_i &= 0 \quad \text{in } \Omega_i, \quad \frac{\partial \chi_i}{\partial \boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n} \quad \text{on } \Gamma_i, \end{aligned}$$

and the function $\tilde{\boldsymbol{u}}$ defined by

$$\tilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{grad}\,\chi_i & \text{in } \Omega_i,\ 0 \leq i \leq I \\ \boldsymbol{0} & \text{in}\,\mathbb{R}^3 \setminus \partial\mathcal{O} \end{cases}$$

which belongs to $\boldsymbol{H}^{\,p}(\text{div},\,\mathbb{R}^3)$ with divergence-free in $\mathbb{R}^3$.

The function $\boldsymbol{\psi}_0 = \mathbf{curl}(\, E * \widetilde{\boldsymbol{u}}\,)$, with E the fundamental solution of the laplacian, satisfies

$$\mathbf{curl}\,\boldsymbol{\psi}_0 = \widetilde{\boldsymbol{u}} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi}_0 = 0 \quad \text{in}\,\mathbb{R}^3.$$

Applying the Calderón Zygmund inequality, we obtain

$$\|\nabla\,\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\Delta\,(E * \widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$$

and $\boldsymbol{\psi}_0|_\Omega$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$.

With a similar proof that to the case $p = 2$, we have

Theorem 4.2 (Tangent Vector Potential)

A function $\boldsymbol{u}$ in $\boldsymbol{H}^{p}(\mathrm{div}, \Omega)$ satisfies (51) if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega, \\ &\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \quad \langle \boldsymbol{\psi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}} = 0,\ 1 \le \mathrm{j} \le \mathrm{J}. \end{aligned} \tag{53}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^{p}(\Omega)}. \tag{54}$$

Theorem 4.3 (Inf-Sup condition in Banach spaces)

Let X and M be two reflexive Banach spaces and X' and M' their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, let $A \in \mathcal{L}(X;\ M')$ and $A' \in \mathcal{L}(M;\ X')$ be the operators defined by

$$\forall v \in X,\ \forall w \in M,\ a(v,w) = < Av, w > = < v, A'w >$$

and $V = \operatorname{Ker} A$. The following statements are equivalent:
i) There exists $\beta > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v,w)}{\|v\|_X \, \|w\|_M} \geq \beta. \tag{55}$$

ii) The operator $A:\ X/V \mapsto M'$ is an isomophism and $1/\beta$ is the continuity constant of A^{-1}.
iii) The operator $A':\ M \mapsto X' \perp V$ is an isomophism and $1/\beta$ is the continuity constant of $(A')^{-1}$.

Proof.

First, we note that $ii) \Leftrightarrow iii)$ because $(X/V)' = X' \perp V$ where this last space contains the elements $f \in X'$ satisfying $\langle f,\, v\rangle = 0$ for any $v \in V$. It suffices then to prove that $i) \Leftrightarrow iii)$. We begin with the implication $i) \Rightarrow iii)$. Due to (55), we deduce that there exists a constant $\beta > 0$ such that:

$$\forall w \in M, \quad \|w\|_M \le \frac{1}{\beta} \sup_{\substack{v \in X \\ v \neq 0}} \frac{|a(v,\, w)|}{\|v\|_X}.$$

So,

$$\|w\|_M \le \frac{1}{\beta} \|A'w\|_{X'}, \tag{56}$$

and A' is injective. Moreover, $\operatorname{Im} A'$ is a closed subspace of X' where $A' :\ M \to X'$. Moreover, $\operatorname{Im} A' = (\operatorname{Ker} A)^{\perp} = X' \perp V$. It remains to prove that $iii) \Rightarrow i)$. For this, it suffices to prove that if $iii)$ holds, then (56) also holds and (55) follows immediately. □

Remark

As consequence, if the Inf-Sup condition (55) is satisfied, then we have the following properties:

i) Because $A' : \ M \mapsto X' \bot V$ is an isomophism, then for any $f \in X'$, satisfying the compatibility condition

$$\forall v \in V, \ < f, v > = 0,$$

there exists a unique $w \in M$ such that

$$\forall v \in X, \ a(v, w) = < f, v > \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta} \|f\|_{X'}. \tag{57}$$

ii) Because $A : \ X/V \mapsto M'$ is an isomophism, then for any $g \in M', \ \exists v \in X$, unique up an additive element of V, such that:

$$\forall w \in M, \ a(v, w) = < g, w > \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta} \|g\|_{M'}$$

We define the space

$$\boldsymbol{V}^p_T(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}^p_T(\Omega);\ \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega \quad \text{and} \\ \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J\},$$

which is a Banach space for the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$.

Lemma 4.4 (Inf Sup Condition)

The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_T(\Omega) \\ \boldsymbol{\varphi} \neq \boldsymbol{0}}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}^{p}_T(\Omega) \\ \boldsymbol{\xi} \neq \boldsymbol{0}}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}^p_T(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{p'}_T(\Omega)}} \geq \beta. \tag{58}$$

Proof. We need the following Helmholtz decomposition: every vector function $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ can be decomposed into a sum

$$\boldsymbol{g} = \nabla \chi + \boldsymbol{z},$$

where

$$\boldsymbol{z} \in \boldsymbol{H}^p(\mathrm{div}, \Omega) \quad \text{with} \quad \mathrm{div}\, \boldsymbol{z} = 0 \quad \text{and} \quad \chi \in W_0^{1,p}(\Omega)$$

with the estimate

$$\|\nabla \chi\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{59}$$

Let $\boldsymbol{\varphi}$ any function of $\boldsymbol{V}_T^{p'}(\Omega)$. We know that

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C \sup_{\substack{\boldsymbol{g}\in\boldsymbol{L}^p(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{\left| \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{g}\, \mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}. \tag{60}$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{i=1}^{I} \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} \nabla q_i^N .$$

So,

$$\widetilde{\boldsymbol{z}} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \widetilde{\boldsymbol{z}} = 0 \quad \text{and} \ \langle \widetilde{\boldsymbol{z}} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0 \quad \forall 0 \leq i \leq I.$$

By Theorem of tangential vector potentiel, there exists a vector potential $\boldsymbol{\psi} \in \boldsymbol{V}_T^p(\Omega)$ such that $\widetilde{\boldsymbol{z}} = \mathbf{curl}\, \boldsymbol{\psi}$ in Ω. This implies that

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \boldsymbol{z} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \widetilde{\boldsymbol{z}} \, \mathrm{d}\boldsymbol{x}.$$

Moreover, we have

$$\begin{aligned} \|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} &\leq \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}| \|\nabla\, q_i^N\|_{\boldsymbol{L}^p(\Omega)} \\ &\leq \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + C\,\|\boldsymbol{z} \cdot \boldsymbol{n}\|_{W^{-\frac{1}{p},p}(\Gamma)}. \end{aligned}$$

Since $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ and $\mathrm{div}\,\boldsymbol{z} = 0$, by using the continuity of the normal trace operator on $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$, (59) and (61) we obtain

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{61}$$

Finally, using Corollary "equivalence of norms" we can write

$$\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}|}{\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}_T^p(\Omega)}},$$

and the Inf-Sup Condition (58) follows immediately from (60).

In the next, we illustrate the importance goal of the Inf-Sup Condition by using it to resolve the following first elliptic system.

Proposition 4.5 (Neumann problem for vector fields)

Assume that $\boldsymbol{v}$ belongs to $\boldsymbol{L}^p(\Omega)$. Then, the following problem

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v}, \quad \operatorname{div}\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0,\ (\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J, \end{cases} \tag{62}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{63}$$

Moreover, if $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{64}$$

Proof.

i) Existence and uniqueness. Thanks to Inf-Sup condition, the following problem: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega), \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \quad (65)$$

satisfies the Inf-Sup condition (58). So, it has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ since the right-hand side defines an element of $(\boldsymbol{V}_T^{p'}(\Omega))'$. By using previous imbeddings results, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Next, we want to extend (65) to any test function $\widetilde{\boldsymbol{\varphi}}$ in $\boldsymbol{X}_T^{p'}(\Omega)$. We consider the solution χ in $W^{1,p'}(\Omega)$ up to an additive constant of the Neumann problem:

$$\Delta\,\chi = \mathrm{div}\,\widetilde{\boldsymbol{\varphi}} \ \text{ in}\,\Omega \quad \text{and} \quad \frac{\partial\,\chi}{\partial \boldsymbol{n}} = 0 \ \text{ on}\,\Gamma. \quad (66)$$

Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle(\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi) \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T. \quad (67)$$

Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}_T^{p'}(\Omega)$. Hence (65) becomes:
Find $\boldsymbol{\xi} \in \boldsymbol{V}_T^{p}(\Omega)$ such that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega), \ \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}} \, \mathrm{d}\boldsymbol{x}. \quad (68)$$

It is easy to proof that every solution of (62) also solves (68). Conversely, let $\boldsymbol{\xi}$ the solution of the problem (68). Then,

$$-\Delta\, \boldsymbol{\xi} = \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{\xi} = \mathbf{curl}\, \boldsymbol{v} \quad \text{in} \quad \Omega.$$

Moreover, since $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}_T^{p}(\Omega)$ we have

$$\operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma \quad \text{and} \quad \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0,$$

for any $1 \leq j \leq J$.

Then, it remains to check the boundary condition of Problem (62):

$$\mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n} \text{ on } \Gamma.$$

But

$$\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{\xi} - \boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega) \quad \text{with} \quad \mathbf{curl}\, \boldsymbol{z} = \mathbf{0} \text{ in } \Omega.$$

Consequently, for any $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega)$ we have:

$$\int_\Omega \boldsymbol{z} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}}\, \mathrm{d}\boldsymbol{x} \;-\; \langle \boldsymbol{z} \times \boldsymbol{n}, \widetilde{\boldsymbol{\varphi}} \rangle_\Gamma = \int_\Omega \mathbf{curl}\, \boldsymbol{z} \cdot \widetilde{\boldsymbol{\varphi}}\, \mathrm{d}\, \boldsymbol{x} = 0.$$

Using (68), we deduce that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega), \quad \langle \boldsymbol{z} \times \boldsymbol{n}, \widetilde{\boldsymbol{\varphi}} \rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ be any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists an element $\widetilde{\boldsymbol{\varphi}}$ of $\boldsymbol{W}^{1,p'}(\Omega)$ such that $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangential component of $\boldsymbol{\mu}$ on Γ. It is clear that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{X}_T^{p'}(\Omega)$ and

$$\langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu} \rangle_\Gamma = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu}_t \rangle_\Gamma = \langle \boldsymbol{z} \times \boldsymbol{n}, \widetilde{\boldsymbol{\varphi}} \rangle_\Gamma = 0.$$

This implies that $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ which is the last boundary condition in (62).

ii) Regularity. Now, we suppose that

$$\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \quad \text{and} \quad \Omega \text{ is of class } \mathcal{C}^{2,1}.$$

Let $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ given by the first step and $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$.
Observe that

$$\boldsymbol{z} \in \boldsymbol{X}^{p}_{N}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega).$$

This implies that

$$\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega).$$

Thanks to the regularity resuts for vector fields, we deduce that

$$\boldsymbol{\xi} \in \boldsymbol{W}^{2,p}(\Omega)$$

and satisfies the estimate (64), which finishes the proof.

Remark

i) Note that we can directly prove the uniqueness of the solution of the problem (62) by using the characterization of the kernels $\boldsymbol{K}^p_T(\Omega)$ and $\boldsymbol{K}^p_N(\Omega)$.

ii) When $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^p(\Omega)$, then

$$(\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}) \times \boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

but neither

$$\mathbf{curl}\,\boldsymbol{\xi} \times \boldsymbol{n} \quad \text{nor} \quad \boldsymbol{v} \times \boldsymbol{n}$$

is defined. However, if $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega)$, then

$$\boldsymbol{v} \times \boldsymbol{n} \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\xi} \times \boldsymbol{n} \quad \text{have a sense in } \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Theorem 4.6 (Normal vector potentials)

$\boldsymbol{u} \in \boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ satisfies:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in}\,\Omega, \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 0 \le j \le J,$$

iff there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{X}^{\,p}(\Omega)$ such that

$$\begin{aligned} \boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega, \\ \boldsymbol{\psi}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \\ \langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ 1 \le i \le I. \end{aligned}$$

This function is unique and moreover, we have:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \le C\|\boldsymbol{u}\,\|_{\boldsymbol{L}^{p}(\Omega)}.$$

Remark : If $\boldsymbol{u} \in \boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)$ then the condition $\langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J$, is necessary and sufficient for the existence of the vector potential $\boldsymbol{\psi}$ satisfying (69) and (69). The condition $\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ 0 \le i \le I$ ensures the uniqueness of $\boldsymbol{\psi}$.

Idea of the proof

- We use the Inf-Sup condition (58) to solve the problem:

Find $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$(\mathcal{P}) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \boldsymbol{n} = 0, \ \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{\psi}_0 \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \end{cases}$$

where $\boldsymbol{\psi}_0 \in \boldsymbol{W}^{1,p}(\Omega)$ is the tangential vector potential.

- Setting $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \mathbf{curl}\, \boldsymbol{\xi}$, we have

  $$\boldsymbol{\psi}_1 \in \boldsymbol{L}^p(\Omega), \quad 0 = \operatorname{div} \boldsymbol{\psi}_1 \in L^p, \quad \mathbf{curl}\, \boldsymbol{\psi}_1 \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{\psi}_1 \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma$$

  Using the regularity results for vector fields, we deduce that $\boldsymbol{\psi}_1 \in \boldsymbol{W}^{1,p}(\Omega)$.

- The required vector potential is given by:

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \sum_{i=1}^{I} \langle \boldsymbol{\psi}_1 \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N$$

Theorem 4.7 (Inf-Sup Condition in X_N^p)

The following Inf-Sup condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_N^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} \geq \beta. \tag{69}$$

Proof The proof is very similar to that of X_T^p. Let $\boldsymbol{\varphi}$ be any function of $\boldsymbol{V}_N^{p'}(\Omega)$. Due to the equivalence norm, we can write: for any $\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)$

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C \sup_{\substack{\boldsymbol{g}\in \boldsymbol{L}^p(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}.$$

We use now the Helmholtz decomposition

$$\boldsymbol{g} = \nabla \chi + \boldsymbol{z}, \quad \text{where } \chi \in W^{1,p}(\Omega) \quad \text{and } \boldsymbol{z} \in \boldsymbol{H}^{p}(\text{div}, \Omega)$$

with

$$\text{div}\, \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and } \boldsymbol{z} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

Moreover, we have the estimate

$$\|\nabla \chi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

The following vector fields

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{j=1}^{J} \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^T$$

satisfies

$$\text{div}\, \widetilde{\boldsymbol{z}} = 0 \ \text{ in } \Omega, \quad \widetilde{\boldsymbol{z}} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \quad \text{and } \langle \widetilde{\boldsymbol{z}} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0 \ \ \forall 1 \leq j \leq J.$$

Using the theorem of normal vector potential the rest of the proof is similar to the Inf-Sup Condtition I.

For Further Reading

C. Amrouche, C. Bernardi, M. Dauge, V. Girault, *Vector potentials in three-dimensional nonsmooth domains*, Math. Meth. Applied Sc., Vol. 21, (1998), 823–864.

C. Amrouche, N. Seloula, *L^p-theory for vector potentials and Sobolev's inequqlities for vector fields. Application to the Stokes problem's with pressure boundary conditions*, M3AS, Volume 23-1, (2013), 37–92.

W. Von Wahl, *Estimating ∇u by* $\operatorname{div} u$, $\mathbf{curl}\, u$, Math. Meth. Appl. Sci., Vol. 15, (1992), 123–143.